Coalgebraic bisimulation

Jan Rutten

CWI & Radboud University

Motivation

- Robin Milner: bisimulation and coinduction.
- Coalgebra, the mathematics of *bisimulation*.
- Behavioural theory of systems.
- CMCS, CALCO; also presence in main conferences.
- Joint work with many persons.

Overview

1. Bisimulation everywhere
2. The power of coinduction
3. More bisimulations, still

1. Bisimulation everywhere

Coalgebraic bisimulation: Aczel & Mendler ’89

An F-bisimulation, for a functor $F : C \to C$:

$$
\begin{array}{ccccc}
X & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & Y \\
\downarrow \alpha & & \exists \downarrow \gamma & & \downarrow \beta \\
F(X) & \xleftarrow[F(\pi_1)]{} & F(R) & \xrightarrow[F(\pi_2)]{} & F(Y)
\end{array}
$$

- As many types of bisimulation as there are functors . . .
- Well-behaved functors: universal coalgebra.
- Bisimulation/Coalgebra = Congruence/Algebra

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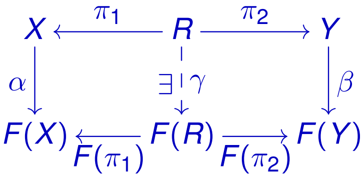

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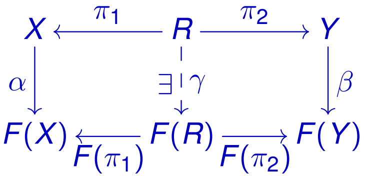

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Example: universes of sets

$$X \xrightarrow{\alpha} \mathcal{P}(X)$$

write $y \in x$ for $y \in \alpha(x)$

- $R \subseteq X \times X$ is a $\mathcal{P}$-bisimulation if for all $(x, y) \in R$:

(1) $\forall\, x' \in x \quad \Rightarrow \quad \exists\, y' \in y \; s.t. \; (x', y') \in R$

and

(2) $\forall\, y' \in y \quad \Rightarrow \quad \exists\, x' \in x \; s.t. \; (x', y') \in R$

- C.f. strong bisimulation on transition systems.

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Sets: examples of bisimulations

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write $y \in x$ for $y \in \alpha(x)$

- E.g., for $x = \{x\}$ and $y = \{y\}$,

  $R = \{ (x, y) \}$ is a bisimulation relation

- E.g., for $x = \{x\}$ and $y = \{x, y\}$,

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- Strong-extensionality

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Example: determinisic automata

$$X \xrightarrow{\langle o, t\rangle} 2 \times X^A$$

(x) is final $\iff$ $o(x) = 1$

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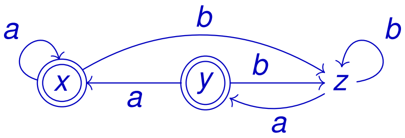

$R = \{ (x, y), (x, x), (y, y), (z, z) \}$ is a bisimulation relation

- Note that here bisimilarity is language (trace) equivalence:

$$L(x) = L(y)$$

. . . which confused the people from CONCUR for a while.

- Cf. ready, failure etc. equivalence [MFPS 2012].

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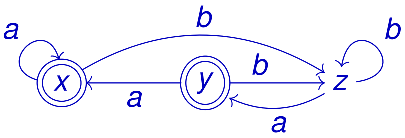

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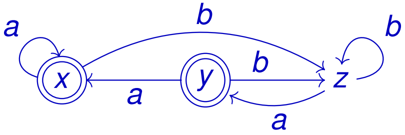

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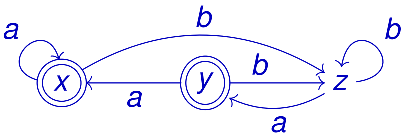

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For a stream $\sigma = (\sigma(0), \sigma(1), \sigma(2), \ldots) \in \mathbb{N}^\omega$,

- initial value: $\sigma(0)$
- derivative: $\sigma' = (\sigma(1), \sigma(2), \sigma(3), \ldots)$

$$\mathbb{N}^\omega \longrightarrow \mathbb{N} \times \mathbb{N}^\omega \qquad \sigma \longmapsto \langle \sigma(0), \sigma' \rangle$$

We call $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ a stream bisimulation if

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$$\sigma \sim \tau \quad \equiv \quad \exists \text{ bisimulation } R \text{ s.t. } \quad \langle\sigma,\tau\rangle \in R$$

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- But first: *defining* streams with

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Stream differential equations

- Recall, for a stream $\sigma = (\sigma(0), \sigma(1), \sigma(2), \ldots) \in \mathbb{N}^\omega$,
  - initial value (= head): $\sigma(0)$
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- Examples of stream differential equations:

initial value	derivative	solution
$\sigma(0) = 1$	$\sigma' = \sigma$	$(1, 1, 1, \ldots)$
$\sigma(0) = 1$	$\sigma' = \sigma + \sigma$	$(2^0, 2^1, 2^2, \ldots)$
$\sigma(0) = 1$	$\sigma' = \sigma \times \sigma$	$(1, 1, 2, 5, 14, 42, \ldots)$

- Existence of unique solutions: by finality!

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A proof by coinduction: Moessner's theorem

- A. Moessner (1951), proof by O. Perron (1951) and I. Paasche (1952).
- Cf. Ralf Hinze: Scans and convolutions - a calculational proof of Moessner's theorem (Oxford University, 2010).
- Our proof: by coinduction (Niqui & R., 2011) . . .
- . . . is a student's exercise.
- Cf. the original proof: advanced binomial coefficient manipulation!!

Moessner's theorem (k = 3)

nat	1	2	3	4	5	6	7	8	9	10	11	12	···
$Drop_3$	1	2		4	5		7	8		10	11	···	
Σ	1	3	7	12	19	27	37	48	···				
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$$\text{nat}^3 = \Sigma \circ \mathit{Drop}_2 \circ \Sigma \circ \mathit{Drop}_3(\text{nat})$$

where nat $= (1, 2, 3, \ldots)$ satisfies

$$\text{nat}(0) = 1 \qquad \text{nat}' = \text{nat} + \text{ones}$$

with ones $= (1, 1, 1, \ldots)$; and

$$\text{nat}^3 = (1^3, 2^3, 3^3, \ldots) = \text{nat} \odot \text{nat} \odot \text{nat}$$

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can all be specified by elementary stream diff. equations.

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Proving Moessner's theorem

$$\text{nat}^3 = \Sigma \circ \mathit{Drop}_2 \circ \Sigma \circ \mathit{Drop}_3(\text{nat})$$

- We use the *coinduction proof principle*: for all $\sigma, \tau \in \mathbb{N}^\omega$,

$$\sigma \sim \tau \quad \Rightarrow \sigma = \tau$$

- So it suffices to construct a bisimulation R with

$$\langle \text{nat}^3, \Sigma \circ \mathit{Drop}_2 \circ \Sigma \circ \mathit{Drop}_3(\text{nat})\rangle \in R$$

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Proving Moessner's theorem

$$\mathrm{nat}^3 = \Sigma \circ Drop_2 \circ \Sigma \circ Drop_3(\,\mathrm{nat}\,)$$

Proof: We define R as the smallest set such that

(i) $\langle \mathrm{nat}^3, \; \Sigma \circ Drop_2 \circ \Sigma \circ Drop_3(\,\mathrm{nat}\,)\rangle \in R$

(ii) $\langle\, \mathrm{nat} \odot (\mathrm{nat} + \mathrm{ones})^2 \,, \; \Sigma \circ Drop_2^0 \circ \Sigma \circ Drop_3^1(\,\mathrm{nat}\,)\,\rangle \in R$

(iii) if $\langle \sigma_1, \sigma_2\rangle \in R$ and $\langle \tau_1, \tau_2\rangle \in R$ then $\langle \sigma_1 + \tau_1, \; \sigma_2 + \tau_2\rangle \in R$

(iv) $\langle \sigma, \sigma\rangle \in R$ (all σ)

Then: R is a bisimulation relation.

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(ii) $\langle \mathrm{nat} \odot (\mathrm{nat} + \mathrm{ones})^2, \Sigma \circ \mathit{Drop}_2^0 \circ \Sigma \circ \mathit{Drop}_3^1(\mathrm{nat})\rangle \in R$

(iii) if $\langle \sigma_1, \sigma_2\rangle \in R$ and $\langle \tau_1, \tau_2\rangle \in R$ then $\langle \sigma_1 + \tau_1, \sigma_2 + \tau_2\rangle \in R$

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- Every functor F has a notion of F-bisimulation . . .
- . . . and F-coinduction definition and proof principles.
- Next: *different* notions of bisimulation for *single* F.
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Different final coalgebra structures on $\mathbb{R}^\omega$

$$\mathbb{R}^\omega \longrightarrow \mathbb{R} \times \mathbb{R}^\omega$$

$$\sigma \longmapsto \langle \sigma(0),\ \sigma' \rangle \qquad \sigma \longmapsto \langle \sigma(0),\ \Delta\sigma \rangle \qquad \sigma \longmapsto \langle \sigma(0),\ \tfrac{d\sigma}{dX} \rangle$$

where

- $\sigma' = (\sigma(1), \sigma(2), \sigma(3), \ldots)$
- $\Delta\sigma = (\sigma(1) - \sigma(0)\,,\ \sigma(2) - \sigma(1)\,,\ \sigma(3) - \sigma(2)\,, \ldots)$
- $\frac{d\sigma}{dX} = (\,1 \cdot \sigma(1)\,,\ 2 \cdot \sigma(2)\,,\ 3 \cdot \sigma(3)\,, \ldots)$

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Non-standard stream calculus

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Non-standard stream differential equations:

initial value	derivative	solution
$\sigma(0) = 1$	$\sigma' = \sigma$	$(1, 1, 1, \ldots)$
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Δ-bisimulation

- $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ is a Δ*-bisimulation* if

$$\forall(\sigma, \tau) \in R : \ \sigma(0) = \tau(0) \text{ and } (\Delta\sigma, \Delta\tau) \in R$$

- We write

$$\sigma \sim_\Delta \tau \equiv \exists \ \Delta\text{-bisimulation } R \text{ s.t. } \langle \sigma, \tau \rangle \in R$$

- Δ-coinduction proof principle:

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An example of Δ-coinduction

- A generalised Euler formula (cf. Taylor series):

$$\sigma = \frac{(\Delta^0\sigma)(0) \times X^0}{(1-X)^1} + \frac{(\Delta^1\sigma)(0) \times X^1}{(1-X)^2} + \frac{(\Delta^2\sigma)(0) \times X^2}{(1-X)^3} + \cdots$$

- *Proof*: Using

$$\Delta\left(\frac{X^{n+1}}{(1-X)^{n+2}}\right) = \frac{X^n}{(1-X)^{n+1}}$$

one easily shows that

$$\{\langle \sigma, sum \rangle \mid \sigma \in \mathbb{R}^\omega\}$$

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Yet another final coalgebra structure on streams

$$\begin{array}{ccc} \mathbb{N}^\omega & & \sigma \\ \downarrow & & \downarrow \\ \mathbb{N} \times \mathbb{N}^\omega \times \mathbb{N}^\omega & & \langle\, \text{head}(\sigma),\ \text{even}(\sigma),\ \text{odd}(\sigma) \,\rangle \end{array}$$

where

- $\text{head}(\sigma) = \sigma(0)$
- $\text{even}(\sigma) = (\sigma(0), \sigma(2), \sigma(4), \ldots)$
- $\text{odd}(\sigma) = (\sigma(1), \sigma(3), \sigma(5), \ldots)$

Final among *zero*-consistent systems

If S is *zero*-consistent:

$$\begin{array}{c} S \\ {\scriptstyle \langle o,l,r\rangle}\big\downarrow \\ \mathbb{N}\times S\times S \end{array} \qquad \forall s\in S,\quad o(l(s))=o(s)$$

then

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- Cf. *Automatic* sequences.

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$\{\text{head}, \text{even}, \text{odd}\}$-differential equations

Ex.

$$\text{head}(\tau) = 0 \qquad \text{even}(\tau) = \tau \qquad \text{odd}(\tau) = \rho$$
$$\text{head}(\rho) = 0 \qquad \text{even}(\rho) = \tau \qquad \text{odd}(\rho) = \tau$$

has as unique solution

$$\tau = 0110100110010110\cdots \qquad \text{Thue-Morse}$$

(and its complement)

- $\{\text{head}, \text{even}, \text{odd}\}$-bisimulation
- $\{\text{head}, \text{even}, \text{odd}\}$-coinduction
- Cf. Kupke & R. [2010,2011].

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Future

1. For instance: classifying SDEs
2. For instance: bisimulation-up-to
3. For instance: . . .

Classifying SDEs

initial value	derivative	solution
$\sigma(0) = 1$	$\sigma' = \sigma$	$(1, 1, 1, \ldots)$
$\sigma(0) = 1$	$\sigma' = \sigma + \sigma$	$(2^0, 2^1, 2^2, \ldots)$
$\sigma(0) = 1$	$\sigma' = \sigma \times \sigma$	$(1, 1, 2, 5, 14, 42, \ldots)$
$\sigma(0) = 1$	$\Delta\sigma = \sigma$	$(2^0, 2^1, 2^2, \ldots)$
$\sigma(0) = 1$	$\frac{d\sigma}{dX} = \sigma$	$(\frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \ldots)$
$\sigma(0) = 0$	$\text{even}(\sigma) = \sigma$ $\text{odd}(\sigma) = \overline{\sigma}$	Thue-Morse